Extension of "Renormalization of period doubling in symmetric four-dimensional volume-preserving maps"

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(Received 14 June 1994)

We numerically reexamine the scaling behavior of period doublings in four-dimensional volume-preserving maps in order to resolve a discrepancy between numerical results on scaling of the coupling parameter and the approximate renormalization results reported by Mao and Greene [Phys. Rev. A **35**, 3911 (1987)]. In order to see the fine structure of period doublings, we extend the simple one-term scaling law to a two-term scaling law. Thus we find a new scaling factor associated with coupling and confirm the approximate renormalization results.

PACS number(s): 05.45.+b, 03.20.+i, 05.70.Jk

Universal scaling behavior of period doubling has been found in area-preserving maps [1–7]. As a nonlinearity parameter is varied, an initially stable periodic orbit may lose its stability and give rise to the birth of a stable period-doubled orbit. An infinite sequence of such bifurcations accumulates at a finite parameter value and exhibits a universal limiting behavior. However, these limiting scaling behaviors are different from those for the one-dimensional dissipative case [8].

An interesting question is whether the scaling results of area-preserving maps carry over higher-dimensional volume-preserving maps. Thus period doubling in four-dimensional (4D) volume-preserving maps has been much studied in recent years [7,9–13]. It has been found in Refs. [11–13] that the critical scaling behaviors of period doublings for two symmetrically coupled area-preserving maps are much richer than those for the uncoupled area-preserving case. There exist an infinite number of critical points in the space of the nonlinearity and coupling parameters. It has been numerically found in [11,12] that the critical behaviors at those critical points are characterized by two scaling factors, δ_1 and δ_2. The value of δ_1 associated with scaling of the nonlinearity parameter is always the same as that of the scaling factor δ ($= 8.721\ldots$) for the area-preserving maps. However, the values of δ_2 associated with scaling of the coupling parameter vary depending on the type of bifurcation routes to the critical points.

The numerical results [11,12] agree well with the approximate analytic renormalization results obtained by Mao and Greene [13], except for the zero-coupling case in which the two area-preserving maps become uncoupled. Using an approximate renormalization method including truncation, they found three relevant eigenvalues, $\delta_1 = 8.9474$, $\delta_2 = -4.4510$, and $\delta_3 = 1.8762$ for the zero-coupling case [14]. However, they believed that the third one, δ_3, is an artifact of the truncation, because only two relevant eigenvalues δ_1 and δ_2 could be identified with the scaling factors numerically found.

In this Brief Report we numerically study the critical behavior at the zero-coupling point in two symmetrically coupled area-preserving maps and resolve the discrepancy between the numerical results on the scaling of the coupling parameter and the approximate renormalization results for the zero-coupling case. In order to see the fine structure of period doublings, we extend the simple one-term scaling law to a two-term scaling law. Thus we find a new scaling factor $\delta_3 = 1.8505\ldots$ associated with coupling, in addition to the previously known coupling scaling factor $\delta_2 = -4.4038\ldots$. The numerical values of δ_2 and δ_3 are close to the renormalization results of the relevant coupling eigenvalues δ_2 and δ_3. Consequently the fixed map governing the critical behavior at the zero-coupling point has two relevant coupling eigenvalues δ_2 and δ_3 associated with coupling perturbations, unlike the cases of other critical points.

Consider a 4D volume-preserving map T consisting of two symmetrically coupled area-preserving Hénon maps [11,12],

$$T: \begin{cases} x_1(t+1) &= -y_1(t) + f(x_1(t)) + g(x_1(t), x_2(t)) \\ y_1(t+1) &= x_1(t) \\ x_2(t+1) &= -y_2(t) + f(x_2(t)) + g(x_2(t), x_1(t)) \\ y_2(t+1) &= x_2(t), \end{cases} \tag{1}$$

where t denotes a discrete time, f is the nonlinear function of the uncoupled Hénon quadratic map [15], i.e.,

$$f(x) = 1 - ax^2, \tag{2}$$

and $g(x_1, x_2)$ is a coupling function obeying a condition

$$g(x, x) = 0 \text{ for any } x. \tag{3}$$

The two-coupled map (1) is called a symmetric map [11,12] because it is invariant under an exchange of coordinates such that $x_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow y_2$. The set of all points, which are invariant under the exchange of coordinates, forms a symmetry plane on which $x_1 = x_2$ and $y_1 = y_2$. An orbit is called an in-phase orbit if it lies on the symmetry plane, i.e., it satisfies

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$$x_1(t) = x_2(t) \equiv x(t), \;\; y_1(t) = y_2(t) \equiv y(t) \;\; \text{for all}\; t. \quad (4)$$

Otherwise it is called an out-of-phase orbit. Here we study only in-phase orbits. They can be easily found from the uncoupled Hénon map because the coupling function g satisfies the condition (3).

Stability analysis of an in-phase orbit can be conveniently carried out [11,12] in a set of new coordinates (X_1, Y_1, X_2, Y_2) defined by

$$X_1 = \frac{(x_1 + x_2)}{2}, \quad Y_1 = \frac{(y_1 + y_2)}{2}, \qquad (5a)$$

$$X_2 = \frac{(x_1 - x_2)}{2}, \quad Y_2 = \frac{(y_1 - y_2)}{2}. \qquad (5b)$$

Note that the in-phase orbit of the map (1) becomes the orbit of the new map (expressed in terms of new coordinates) with $X_2 = Y_2 = 0$. Moreover the new coordinates X_1 and Y_1 of the in-phase orbit also satisfy the uncoupled Hénon map.

Linearizing the new map at an in-phase orbit point, we obtain the Jacobian matrix J which decomposes into two 2×2 matrices [11,12]:

$$J = \begin{pmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & J_2 \end{pmatrix}. \qquad (6)$$

Here $\mathbf{0}$ is the 2×2 null matrix, and

$$J_1 = \begin{pmatrix} f'(X_1) & -1 \\ 1 & 0 \end{pmatrix}, \qquad (7)$$

$$J_2 = \begin{pmatrix} f'(X_1) - 2G(X_1) & -1 \\ 1 & 0 \end{pmatrix}, \qquad (8)$$

where $f'(X) = \frac{df}{dX}$ and $G(X) \equiv \left.\frac{\partial g(X_1,X_2)}{\partial X_2}\right|_{X_1=X_2=X}$. Hereafter the function $G(X)$ will be called the "reduced" coupling function of $g(X_1, X_2)$. Note also that the determinant of each 2×2 matrix J_i $(i = 1, 2)$ is one, i.e., $\mathrm{Det}(J_i) = 1$. Hence they are area-preserving maps.

Stability of an in-phase orbit with period q is then determined from the q product M_i of the 2×2 matrix J_i:

$$M_i \equiv \prod_{t=0}^{q-1} J_i(X_1(t)), \quad i = 1, 2. \qquad (9)$$

Since $\mathrm{Det}(M_i) = 1$, each matrix M_i has a reciprocal pair of eigenvalues, λ_i and λ_i^{-1}. Associate with a pair of eigenvalues $(\lambda_i, \lambda_i^{-1})$ a stability index [16],

$$\rho_i = \lambda_i + \lambda_i^{-1}, \quad i = 1, 2 \qquad (10)$$

which is just the trace of M_i, i.e., $\rho_i = \mathrm{Tr}(M_i)$. Since M_i is a real matrix, ρ_i is always real. Note that the first stability index ρ_1 is just that for the case of the uncoupled Hénon map and hence coupling affects only the second stability index ρ_2.

An in-phase orbit is stable only when the moduli of its stability indices are less than or equal to two, i.e., $|\rho_i| \leq 2$ for $i = 1$ and 2. A period-doubling (tangent) bifurcation occurs when each stability index ρ_i decreases (increases) through -2 (2). Hence the stable region of the in-phase orbit in the parameter plane is bounded by four bifurcation lines associated with tangent and period-doubling bifurcations (i.e., those curves determined by the equations $\rho_i = \pm 2$ for $i = 0, 1$). When the stability index ρ_1 decreases through -2, the in-phase orbit loses its stability via in-phase period-doubling bifurcation and gives rise to the birth of the period-doubled in-phase orbit. Here we are interested in scaling behaviors of such in-phase period-doubling bifurcations.

As an example we consider a linearly coupled case in which the coupling function is

$$g(x_1, x_2) = \frac{c}{2}(x_2 - x_1). \qquad (11)$$

Here c is a coupling parameter. As previously observed in Refs. [11,12], each "mother" stability region bifurcates into two "daughter" stability regions successively in the parameter plane. Thus the stable regions of in-phase orbits of period 2^n $(n = 0, 1, 2, \ldots)$ form a "bifurcation" tree in the parameter plane [17].

An infinite sequence of connected stability branches (with increasing period) in the bifurcation tree is called a bifurcation "route" [11,12]. Each bifurcation route can be represented by its address, which is an infinite sequence of two symbols (e.g., L and R). A "self-similar" bifurcation "path" in a bifurcation route is formed by following a sequence of parameters (a_n, c_n), at which the in-phase orbit of level n (period 2^n) has some given stability indices (ρ_1, ρ_2) (e.g., $\rho_1 = -2$ and $\rho_2 = 2$) [11,12]. All bifurcation paths within a bifurcation route converge to an accumulation point (a^*, c^*), where the value of a^* is always the same as that of the accumulation point for the area-preserving case (i.e., $a^* = 4.136\,166\,803\,904\ldots$), but the value of c^* varies depending on the bifurcation routes. Thus each bifurcation route ends at a critical point (a^*, c^*) in the parameter plane.

It has been numerically found that scaling behaviors near a critical point are characterized by two scaling factors, δ_1 and δ_2 [11,12]. The value of δ_1 associated with scaling of the nonlinearity parameter is always the same as that of the scaling factor δ $(= 8.721\ldots)$ for the area-preserving case. However, the values of δ_2 associated with scaling of the coupling parameter vary depending on the type of bifurcation routes. These numerical results agree well with analytic renormalization results [13], except for the case of one specific bifurcation route, called the E route. The address of the E route is $[(L, R,)^\infty]$ $(\equiv [L, R, L, R, \ldots])$ and it ends at the zero-coupling critical point $(a^*, 0)$.

Using an approximate renormalization method including truncation, Mao and Greene [13] obtained three relevant eigenvalues, $\delta_1 = 8.9474$, $\delta_2 = -4.4510$, and $\delta_3 = 1.8762$ for the zero-coupling case; hereafter the two eigenvalues δ_2 and δ_3 associated with coupling will be called the coupling eigenvalues (CE's). The two eigenvalues δ_1 and δ_2 are close to the numerical results of the nonlinearity-parameter scaling factor $\delta_1 (= 8.721\ldots)$ and the coupling-parameter scaling factor $\delta_2 (= -4.403\ldots)$ for the E route. However, they believed that the second relevant CE δ_3 is an artifact of the truncation, because it could not be identified with anything obtained by a direct numerical method.

In order to resolve the discrepancy between the numerical results and the renormalization results for the zero-coupling case, we numerically reexamine the scaling behavior associated with coupling. Extending the simple one-term scaling law to a two-term scaling law, we find a new scaling factor $\delta_3 = 1.8505\ldots$ associated with coupling in addition to the previously found coupling scaling factor $\delta_2 = -4.4038\ldots$, as will be seen below. The values of these two coupling scaling factors are close to the renormalization results of the relevant CE's δ_2 and δ_3.

We follow the in-phase orbits of period 2^n up to level $n = 14$ in the E route and obtain a self-similar sequence of parameters (a_n, c_n), at which the pair of stability indices, $(\rho_{0,n}, \rho_{1,n})$, of the orbit of level n is $(-2, 2)$. The scalar sequences $\{a_n\}$ and $\{c_n\}$ converge geometrically to their limit values, a^* and 0, respectively. In order to see their convergence, define $\delta_n \equiv \Delta a_{n+1}/\Delta a_n$ and $\mu_n \equiv \Delta c_{n+1}/\Delta c_n$, where $\Delta a_n = a_n - a_{n-1}$ and $\Delta c_n = c_n - c_{n-1}$. Then they converge to their limit values δ and μ as $n \to \infty$, respectively. Hence the two sequences $\{\Delta a_n\}$ and $\{\Delta c_n\}$ obey one-term scaling laws asymptotically:

$$\Delta a_n = C^{(a)}\delta^{-n}, \quad \Delta c_n = C^{(c)}\mu^{-n} \quad \text{for large } n, \tag{12}$$

where $C^{(a)}$ and $C^{(c)}$ are some constants, $\delta = 8.721\ldots$, and $\mu = -4.403\ldots$. The values of δ and μ are close to the renormalization results of the first and second relevant eigenvalues δ_1 and δ_2, respectively.

In order to take into account the effect of the second relevant CE δ_3 on the scaling of the sequence $\{\Delta c_n\}$, we extend the simple one-term scaling law (12) to a two-term scaling law:

$$\Delta c_n = C_1\mu_1^{-n} + C_2\mu_2^{-n} \quad \text{for large } n, \tag{13}$$

where $|\mu_1| > |\mu_2|$. This is a kind of multiple scaling law [18]. Equation (13) gives

$$\Delta c_n = t_1\Delta c_{n+1} - t_2\Delta c_{n+2}, \tag{14}$$

where $t_1 = \mu_1 + \mu_2$ and $t_2 = \mu_1\mu_2$. Then μ_1 and μ_2 are solutions of the following quadratic equation:

$$\mu^2 - t_1\mu + t_2 = 0. \tag{15}$$

To evaluate μ_1 and μ_2, we first obtain t_1 and t_2 from Δc_n's using Eq. (14):

$$t_1 = \frac{\Delta c_n \Delta c_{n+1} - \Delta c_{n-1}\Delta c_{n+2}}{\Delta c_{n+1}^2 - \Delta c_n \Delta c_{n+2}}, \tag{16a}$$

$$t_2 = \frac{\Delta c_n^2 - \Delta c_{n+1}\Delta c_{n-1}}{\Delta c_{n+1}^2 - \Delta c_n \Delta c_{n+2}}. \tag{16b}$$

Note that Eqs. (13)–(16b) hold only for large n. In fact the values of t_i's and μ_i's $(i = 1, 2)$ depend on the level n. Therefore we explicitly denote t_i's and μ_i's by $t_{i,n}$'s and $\mu_{i,n}$'s, respectively. Then each of them converges to a constant as $n \to \infty$:

$$\lim_{n\to\infty} t_{i,n} = t_i, \quad \lim_{n\to\infty} \mu_{i,n} = \mu_i, \quad i = 1, 2. \tag{17}$$

TABLE I. Scaling factors $\mu_{1,n}$ and $\mu_{2,n}$ in the two-term scaling for the coupling parameter are shown in the second and third columns, respectively. A product of them, $\mu_{1,n}^2/\mu_{2,n}$, is shown in the fourth column.

n	$\mu_{1,n}$	$\mu_{2,n}$	$\frac{\mu_{1,n}^2}{\mu_{2,n}}$
5	−4.403 908 128	10.437 4	1.858 17
6	−4.403 899 694	10.465 9	1.853 09
7	−4.403 898 736	10.458 2	1.854 46
8	−4.403 897 867	10.474 8	1.851 52
9	−4.403 897 847	10.473 9	1.851 68
10	−4.403 897 806	10.478 4	1.850 89
11	−4.403 897 807	10.478 6	1.850 85
12	−4.403 897 805	10.479 7	1.850 65

Three sequences $\{\mu_{1,n}\}$, $\{\mu_{2,n}\}$, and $\{\mu_{1,n}^2/\mu_{2,n}\}$ are shown in Table I. The second column shows rapid convergence of $\mu_{1,n}$ to its limit values μ_1 $(= -4.403\,897\,805)$, which is close to the renormalization result of the first relevant CE (i.e., $\delta_2 = -4.4510$). From the third and fourth columns, we also find that the second scaling factor μ_2 is given by a product of two relevant CE's δ_2 and δ_3,

$$\mu_2 = \frac{\delta_2^2}{\delta_3}, \tag{18}$$

where $\delta_2 = \mu_1$ and $\delta_3 = 1.850\,65$. It has been known that every scaling factor in the multiple-scaling expansion of a parameter is expressed by a product of the eigenvalues of a linearized renormalization operator [18]. Note that the value of δ_3 is close to the renormalization result of the second relevant CE (i.e., $\delta_3 = 1.8762$).

We now study the coupling effect on the second stability index $\rho_{2,n}$ of the in-phase orbit of period 2^n near the zero-coupling critical point $(a^*, 0)$. Figure 1 shows three plots of $\rho_{2,n}(a^*, c)$ versus c for $n = 4, 5$, and 6. For $c = 0$, $\rho_{2,n}$ converges to a constant ρ_2^* $(= -2.543\,510\,20\ldots)$, called the critical stability index [12], as $n \to \infty$. However, when c is nonzero $\rho_{2,n}$ diverges as $n \to \infty$, i.e.,

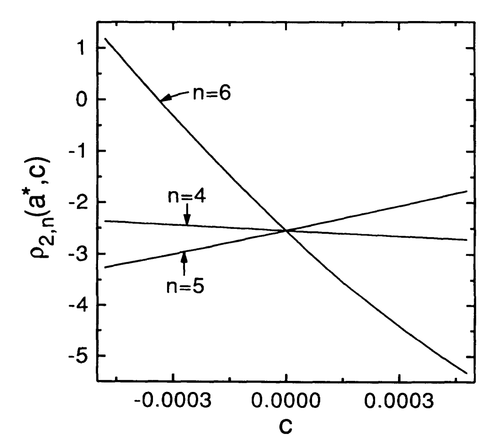

FIG. 1. Plots of the second stability index $\rho_{2,n}(a^*, c)$ versus c for $n = 4, 5, 6$.

its slope S_n ($\equiv \frac{\partial \rho_{2,n}}{\partial c}\Big|_{(a^*,0)}$) at the zero-coupling critical point diverges as $n \to \infty$.

The sequence $\{S_n\}$ obeys a two-term scaling law,

$$S_n = D_1\nu_1^n + D_2\nu_2^n \quad \text{for large } n, \tag{19}$$

where $|\nu_1| > |\nu_2|$. This equation gives

$$S_{n+2} = r_1 S_{n+1} - r_2 S_n, \tag{20}$$

where $r_1 = \nu_1 + \nu_2$ and $r_2 = \nu_1\nu_2$. As in the scaling for the coupling parameter, we first obtain r_1 and r_2 of level n from S_n's:

$$r_{1,n} = \frac{S_{n+1}S_n - S_{n+2}S_{n-1}}{S_n^2 - S_{n+1}S_{n-1}}, \quad r_{2,n} = \frac{S_{n+1}^2 - S_nS_{n+2}}{S_n^2 - S_{n+1}S_{n-1}}. \tag{21}$$

Then the scaling factors $\nu_{1,n}$ and $\nu_{2,n}$ of level n are given by the roots of the quadratic equation, $\nu_n^2 - r_{1,n}\nu_n + r_{2,n} = 0$. They are listed in Table II and converge to constants ν_1 $(= -4.403\,897\,805\,09)$ and ν_2 $(= 1.850\,535)$ as $n \to \infty$, whose accuracies are higher than those of the coupling-parameter scaling factors. Note that the values of ν_1 and ν_2 are also close to the renormalization results of the two relevant CE's δ_2 and δ_3.

TABLE II. Scaling factors $\nu_{1,n}$ and $\nu_{2,n}$ in the two-term scaling for the slope of the second stability index are shown.

n	$\nu_{1,n}$	$\nu_{2,n}$
5	−4.403 898 453 59	1.851 433 5
6	−4.403 897 730 29	1.850 782 6
7	−4.403 897 813 85	1.850 603 6
8	−4.403 897 804 07	1.850 553 8
9	−4.403 897 805 21	1.850 540 0
10	−4.403 897 805 07	1.850 536 1
11	−4.403 897 805 09	1.850 535 0
12	−4.403 897 805 09	1.850 534 9

We have also studied several other coupling cases with the coupling function, $g(x_1, x_2) = \frac{c}{2}(x_2^n - x_1^n)$ (n is a positive integer). In all cases studied ($n = 2, 3, 4, 5$), the scaling factors of both the coupling parameter c and the slope of the second stability index ρ_2 are found to be the same as those for the above linearly coupled case ($n = 1$) within numerical accuracy. Hence universality also seems to be well obeyed.

This work was supported by the Basic Science Research Institute Program, Ministry of Education, Korea, 1994, Project No. BSRI-94-2401.

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